Resit exam LA I, 18 April 2012

1. (a) $\underset{\sim}{x} = \underset{\sim}{0}$ is always a solution of $A\underset{\sim}{x} = \underset{\sim}{0}$. There are then no $\alpha \in \mathbb{R}$ for which $A\underset{\sim}{x} = \underset{\sim}{0}$ is inconsistent. $\boxed{2}$

(b) transform A to row Echelon form:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 3 \\ 2 & -2 & \alpha \end{pmatrix} \xrightarrow[\text{III} \to \text{III} - 2\text{I}]{\text{II} \to \text{II} + \text{I}} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 6 & 4 \\ 0 & -6 & \alpha - 2 \end{pmatrix}$$

$$\xrightarrow{\text{III} \to \text{III} + \text{II}} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 6 & 4 \\ 0 & 0 & \alpha + 2 \end{pmatrix}$$ $\boxed{4}$

Hence: for $\alpha \neq -2$ there is exactly one solution $\underset{\sim}{x} = 0$

(c) For $\alpha = -2$, there are infinitely many solutions

Let $x_3 = \beta \in \mathbb{R}$

$\Rightarrow x_2 = -\frac{2}{3}\beta$, $x_1 = -\beta + \frac{4}{3}\beta = \frac{1}{3}\beta$ $\boxed{4}$

$\Rightarrow$ solution set $= \left\{ \beta \begin{pmatrix} 1/3 \\ -2/3 \\ 1 \end{pmatrix} : \beta \in \mathbb{R} \right\}$

(d) transform augmented matrix $(A|\underset{\sim}{b})$ to row Echelon form

$$(A|\underset{\sim}{b}) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & 4 & 3 & 0 \\ 2 & -2 & \alpha & 1 \end{array}\right) \xrightarrow[\text{III} \to \text{III} - 2\text{I}]{\text{II} \to \text{II} + \text{I}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 6 & 4 & 0 \\ 0 & -6 & \alpha - 2 & 1 \end{array}\right)$$

$$\xrightarrow{\text{III} \to \text{III} + \text{II}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 6 & 4 & 0 \\ 0 & 0 & \alpha + 2 & 1 \end{array}\right)$$ $\boxed{4}$

Hence: $A\underset{\sim}{x} = \underset{\sim}{b}$ is inconsistent for $\alpha = -2$

(e) For $\alpha \neq -2$: $x_3 = \frac{1}{\alpha+2}$, $x_2 = -\frac{2}{3}x_3 = -\frac{2}{3\alpha+6}$

$$x_1 = -\frac{1}{\alpha+2} + \frac{4}{3}\frac{1}{\alpha+2} = \frac{1}{3\alpha+6}$$ $\boxed{4}$

2. (a) Proof: Suppose there is another 0-element. Call it $\underset{\sim}{0}'$, i.e.

$$\underset{\sim}{v} = \underset{\sim}{0}' + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{0}' \quad \text{for all } \underset{\sim}{v} \in V$$ $\boxed{4}$

Now. $\underset{\sim}{0}' = \underset{\sim}{0}' + \underset{\sim}{0}$ because $\underset{\sim}{0}$ is a $\underset{\sim}{0}$-element

$= \underset{\sim}{0}$ because $\underset{\sim}{0}'$ is a 0-element.

Hence $\underset{\sim}{0}' = \underset{\sim}{0}$. The 0-element is thus unique. $\square$

(b) Proof. Let $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n \in V$ where one of the $\underset{\sim}{v}_k$ is equal to $\underset{\sim}{0}$.

Without restriction $\underset{\sim}{v}_n = \underset{\sim}{0}$ (else relabel).

$\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n$ linearly dependent means that there are $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $\alpha_1, \ldots, \alpha_n$ not all equal to zero and that $\boxed{4}$

$$\alpha_1 \underset{\sim}{v}_1 + \ldots + \alpha_n \underset{\sim}{v}_n = \underset{\sim}{0}. \quad (*)$$

However, since $\underset{\sim}{v}_n = \underset{\sim}{0}$ Eq (*) always solution

$\alpha_1, \ldots, \alpha_{n-1} = 0$ and $\alpha_n \in \mathbb{R}$ arbitrary.

$\Rightarrow \underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n$ linearly dependent. $\square$

(c) Proof: Let $\underset{\sim}{u}_1 = \underset{\sim}{v}_1 + \underset{\sim}{v}_2$, $\underset{\sim}{u}_2 = \underset{\sim}{v}_2 + \underset{\sim}{v}_3$, $\underset{\sim}{u}_3 = \underset{\sim}{v}_3 + \underset{\sim}{v}_1$

and (*) $\alpha_1 \underset{\sim}{u}_1 + \alpha_2 \underset{\sim}{u}_2 + \alpha_3 \underset{\sim}{u}_3 = \underset{\sim}{0}$ with $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

Show: $\alpha_1 = \alpha_2 = \alpha_3 = \underset{\sim}{0}$.

We have $(*) \Leftrightarrow \alpha_1 \underset{\sim}{v}_1 + \alpha_1 \underset{\sim}{v}_2 + \alpha_2 \underset{\sim}{v}_2 + \alpha_2 \underset{\sim}{v}_3 + \alpha_3 \underset{\sim}{v}_3 + \alpha_3 \underset{\sim}{v}_1 = \underset{\sim}{0}$

$\Leftrightarrow \quad (\alpha_1 + \alpha_3) \underset{\sim}{v}_1 + (\alpha_1 + \alpha_2) \underset{\sim}{v}_2 + (\alpha_2 + \alpha_3) \underset{\sim}{v}_3 = \underset{\sim}{0}$

$\Rightarrow \alpha_1 + \alpha_3 = 0, \ \alpha_1 + \alpha_2 = 0, \ \alpha_2 + \alpha_3 = 0$

because $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$ linear. independ.

$$\Leftrightarrow \quad \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{=: A} \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}}_{=: \underset{\sim}{\alpha}} = \underset{\sim}{0}$$

$A \underset{\sim}{\alpha} = \underset{\sim}{0}$ has solution $\underset{\sim}{\alpha} \neq \underset{\sim}{0}$ only if A is singular.

Row Echelon form of A:

$$A \xrightarrow{II \to II - I} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{III \to III - II} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$\boxed{4}$

$\Rightarrow$ A is not singular.

$\Rightarrow \underset{\sim}{\alpha} = \underset{\sim}{0}$ and hence $\underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{u}_3$ are linearly independent. $\square$

3. (a) Let $A, B \in L$, $\alpha, \beta \in \mathbb{R}$

$\Rightarrow \alpha A + \beta B$ has components $\alpha a_{ij} + \beta b_{ij}$

We have $a_{ij} = - a_{ji}$ and $b_{ij} = - b_{ji}$ $\boxed{4}$

because A and B are anti symmetric

$\Rightarrow \alpha a_{ji} + \beta b_{ji} = -(\alpha a_{ij} + \beta b_{ij}) \quad \forall i,j$

$\Rightarrow (\alpha A + \beta B)^T = -(\alpha A + \beta B)$

$\Rightarrow \alpha A + \beta B \in L$. $\square$

(b) $L = \left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$ has basis $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$.

$\boxed{4}$

L has dimension 1.

(c) $S = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$ has basis

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

$\boxed{4}$

S has dimension 3

(d) Let $A \in \mathbb{R}^{2 \times 2}$

$\Rightarrow \quad A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$

Set $A_2 = \frac{1}{2}\left(A + A^T\right)$ is symmetric

$A_1 = \frac{1}{2}\left(A - A^T\right)$ is antisymmetric

$\boxed{4}$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \frac{1}{2}\left(A + A^T\right) = \frac{1}{2}\begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}$

$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(b+c)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\frac{1}{2}\left(A - A^T\right) = \frac{1}{2}\begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}$

$= \frac{1}{2}(b-c)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

4. (a) elements of P_n are of the form

$a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}$,

with $a_0, \ldots, a_{n-1} \in \mathbb{R}$

$\boxed{4}$

$\Rightarrow P_n$ has basis $\{1, x, x^2, \ldots, x^{n-1}\}$

(b) P_n has dimension n.

$\boxed{2}$

(c) Let $p, q \in P_4$ and $\alpha \in \mathbb{R}$

$$\Rightarrow T\big(p(x)+q(x)\big) = \big(p(x)+q(x)\big) - x\big(p(x)+q(x)\big)'$$
$$= p(x)+q(x) - xp'(x) - xq'(x)$$
$$= p(x) - xp'(x) + q(x) - xq'(x) \qquad \boxed{3}$$
$$= T\big(p(x)\big) + T\big(q(x)\big)$$

$$T\big(\alpha p(x)\big) = \alpha p(x) - x\big(\alpha p(x)\big)'$$
$$= \alpha\big(p(x) - xp'(x)\big) = \alpha T\big(p(x)\big)$$

(d) Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$\Rightarrow T\big(p(x)\big) = p(x) - xp'(x)$$
$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \qquad \boxed{4}$$
$$- x\left(a_1 + 2a_2 x + 3a_3 x^2\right)$$
$$= a_0 - a_2 x^2 - 2a_3 x^3$$
$$= 0 \quad \text{for } a_0 = a_2 = a_3 = 0$$
and $a_1 \in \mathbb{R}$ arbitrary

$$\Rightarrow \ker T = \{ax : a \in \mathbb{R}\}$$

(e) $[1]_E = (1,0,0,0)^T$, $[x]_E = (0,1,0,0)^T$, $[x^2]_E = (0,0,1,0)^T$,
$[x^3]_E = (0,0,0,1)^T$

$T(1) = 1$, $\quad [T(1)]_E = (1,0,0,0)^T$

$T(x) = 0$, $\quad [T(x)]_E = (0,0,0,0)^T$ $\qquad \boxed{4}$

$T(x^2) = x^2 - 2x^2 = -x^2$, $\quad [T(x^2)]_E = (0,0,-1,0)^T$

$T(x^3) = x^3 - 3x^3 = -2x^3$, $\quad [T(x^3)]_E = (0,0,0,-2)^T$

$$\Rightarrow [T]_E =: A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

(f) T has rank 3 because the columns of $[T]_E$ span a 3-dimensional space. $\boxed{3}$

5. (a) Let $\underset{\sim}{x}, \underset{\sim}{y} \in W^\perp$ and $\alpha, \beta \in \mathbb{R}$

$$\Rightarrow \underset{\sim}{x}^T \underset{\sim}{z} = \underset{\sim}{y}^T \underset{\sim}{z} = 0 \qquad \forall z \in W$$

$$(\alpha \underset{\sim}{x} + \beta \underset{\sim}{y})^T = \alpha \underset{\sim}{x}^T + \beta \underset{\sim}{y}^T$$ $\boxed{4}$

Hence

$$(\alpha \underset{\sim}{x} + \beta \underset{\sim}{y})^T \underset{\sim}{z} = \alpha \underset{\sim}{x}^T \underset{\sim}{z} + \beta \underset{\sim}{y}^T \underset{\sim}{z} = 0$$

$$\Rightarrow \alpha \underset{\sim}{x} + \beta \underset{\sim}{y} \in W^\perp$$

(b) Let $\underset{\sim}{x} = (x_1, \dots, x_n)^T \in W \cap W^\perp$

$$\Rightarrow \underset{\sim}{x}^\perp \underset{\sim}{x} = 0$$ $\boxed{4}$

$$\shortparallel \; x_1^2 + x_2^2 + \dots + x_n^2$$

$$\Rightarrow x_1 = x_2 = \dots = x_n = 0 \Rightarrow \underset{\sim}{x} = 0$$

(c) $W^\perp = N(A^T)$, $\mathbb{R}^n = W \oplus W^\perp$

$$\Rightarrow \dim W^\perp = n - \dim W$$ $\boxed{4}$

We know $\ker(A^T) \oplus R(A) = \mathbb{R}^n$

and $W = R(A)$

$\Rightarrow W^\perp = N(A^T)$ because direct sum is unique

6. (a) $p(\lambda) = |A - \lambda\, id| = \begin{vmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 4 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 4\lambda$ $\boxed{4}$

(b) $p(\lambda) = 0 \Leftrightarrow \lambda = 0$ or $\lambda^2 = 4$

$\Leftrightarrow \lambda = 0$ or $\lambda = 2$ or $\lambda = -2$ $\boxed{3}$

Eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = -2$

(c) eigenvector for $\lambda_1 = 0$: $(A - 0\, id)\, \underset{\sim}{v_1} = \underset{\sim}{0}$

$\Leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underset{\sim}{0}$

let $x_3 = 1 \Rightarrow x_1 = -4$, $x_2 = 0$ $\Rightarrow \underset{\sim}{v_1} = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$

eigenvector for $\lambda_2 = 2$: $(A - 2\, id)\, \underset{\sim}{v_2} = \underset{\sim}{0}$

$\Leftrightarrow \begin{pmatrix} -2 & 0 & 0 \\ 1 & -2 & 4 \\ 0 & 1 & -2 \end{pmatrix} \underset{\sim}{v_2} = \underset{\sim}{0}$

$II \to II + \frac{1}{2} I$
$\Leftrightarrow \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -2 \end{pmatrix} \underset{\sim}{v_2} = \underset{\sim}{0}$

$III \to III + \frac{1}{2} II$
$\Leftrightarrow \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \underset{\sim}{v_2} = 0$

let $x_3 = 1 \Rightarrow x_2 = 2$, $x_1 = 0$ $\Rightarrow \underset{\sim}{v_2} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

eigenvector for $\lambda_3 = -2$: $(A + 2\, id)\, \underset{\sim}{v_3} = \underset{\sim}{0}$

$\Leftrightarrow \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix} \underset{\sim}{v_3} = \underset{\sim}{0}$

$II \to II - \frac{1}{2} I$
$\Leftrightarrow \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix} \underset{\sim}{v} = \underset{\sim}{0}$

$III \to III - \frac{1}{2} II$
$\Leftrightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \underset{\sim}{v} = \underset{\sim}{0}$

Let $x_3 = 1 \Rightarrow x_2 = -2 \quad , \quad x_1 = 0$

$\boxed{6}$

$$\text{eigenvector } \underset{\sim}{V_3} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

(d) Let $T = \begin{pmatrix} \underset{\sim}{V_1} , \underset{\sim}{V_2} , \underset{\sim}{V_3} \end{pmatrix} = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & -2 \\ 1 & 1 & 1 \end{pmatrix}$

$\boxed{3}$

$$\Rightarrow T^{-1} A T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$